Replacement of the Riemann Sphere by a Horn Torus (‘doughnut’) and Conformal Mappings between Plane, Sphere and Horn Torus

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Preliminary Remark: This publication doesn't require particular mathematical skills, is no real contribution to scientific cognition and seems to have no great relevance in science. But the geometrical figure horn torus is unduly underrepresented in serious publications, that it is justified to mention it in as much contexts as possible. After all, the conformal mapping, that we discuss here, and the bijectivity between horn torus and Riemann sphere distinguishes it as a member of complex manifolds.

Wherever you apply the Riemann sphere in theoretical physics, you may replace it by the horn torus and instantaneously gain a lot more properties of the described object. And by *dynamization*, i.e. adding revolutions, rotations and changes of size, and after intense familiarization with then detected new multifarious and sophisticated properties, there will open a completely new mathematical world which bears the possibility to see and describe the fundaments of our 'real world' in a quite different and potentially most comprehensive way.

Abstract: For certain applications, especially regarding fundamental physical questions, it might be useful to replace the well-known and well-established Riemann sphere by the geometric figure horn torus, which comprises much more properties, complexity and capabilities than the sphere provides. In this paper we will describe the analytical method for respective conformal mappings.

Key Words: horn torus model, conformal mapping, Riemann sphere, compactification, complex analysis, manifolds, infinity

Acknowledgement: I want to thank Vyacheslav Puha, Murmansk, Russia, for the idea and the stimulus to search for this conformal mappings and Professor Saburou Saitoh, Kiryu, Japan for the kind inclusion of the topic into several of his papers and presentations.

History: horn torus idea and distribution as private letters 1988, first prints of the original texts in German (DornTorus 1+2) 1996 and 1998, website dorntorus.de in German 2000, website horntorus.com 2006, continuously enhanced since then, this subject 'conformal mapping' 2018 in correspondence with Vyacheslav Puha and Professor Saburou Saitoh.

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1 Riemannian Stereographic Projection

There surely is no need to discuss the method and properties of mapping points, lines and curves (functions) from the complex plane onto the Riemannian unit sphere. The stereographic projection with all its properties and associated mathematical laws is sufficiently known. Here we only designate the following elements for figure 1:

M:	centre of the Riemann sphere
0:	touch point of plane and sphere (zero point of complex plane)
N:	'north pole' of sphere on extended line $\overline{0M}$ (opposite 0)
Z:	point on complex plane, assigned to the complex number z
$\lvert z \rvert$:	absolute value of z *and* length of line $\overline{0Z}$
P:	intersection point of line $\overline{NZ}$ on surface of sphere
α:	central angle 0MP for chord $\overline{0P}$

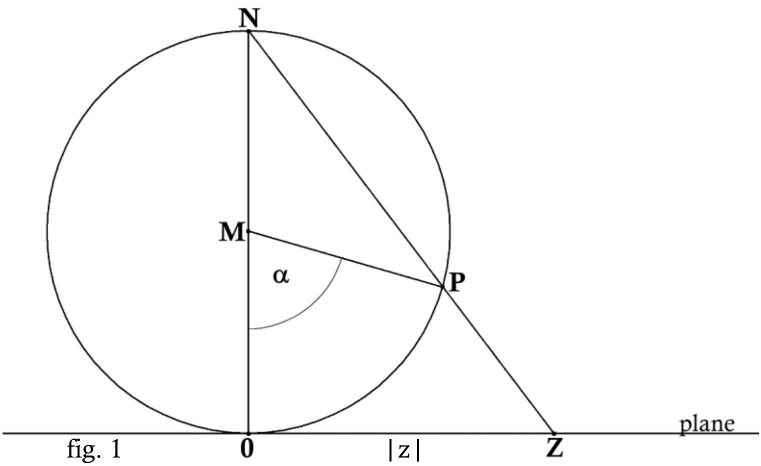

fig. 1

Because of angle 0NZ = $\alpha/2$ (it is circumferential angle for chord $\overline{0P}$) and $\overline{0N} = 1$ (by definition of Riemann sphere) we have for the mappings from plane to sphere and vice versa:

$$\alpha = 2 \tan^{-1}(\lvert z \rvert) \tag{1.1}$$

$$\lvert z \rvert = \tan(\alpha/2) \tag{1.2}$$

2 Mapping from Sphere to Horn Torus and vice versa

A similar stereographic projection like the Riemannian probably does not exist for these mappings, when conformality likewise is stipulated, and so we study sphere and horn torus separately, not for the representation of complex numbers only but for all cases and generally valid.

Therefor we don't declare any appropriate stereographic projection and try to proof the conformality afterwards but we use the conditions of conformality instead to compile and establish the wanted mapping analytically:

One condition for conformal mapping is that small circles on the origin are mapped as small circles on the target surface and therefore we construct small circles on both surfaces that only are dependent on the position of points P on the surface, i.e. only dependent on the angles $\alpha = \angle NMP$ for the sphere (see fig. 2) resp. $\varphi = \angle SMP$ for the horn torus (fig. 3). Due to rotation symmetry of both figures the rotation angle ω doesn't play any role and is arbitrary, and likewise, because of mirror symmetry, α can be exchanged by $\pi - \alpha$ (what corresponds to the exchange of north and south pole of the sphere) and φ can be exchanged by $2\pi - \varphi$. Even the radii r of both figures turn out to be arbitrary and independent.

Following drawings show longitudes, spacing 10°, and latitudes, spacing 20°, P located at angles $\alpha = 50°$ resp. $\varphi = 140°$, $\omega = 35°$ on both figures.

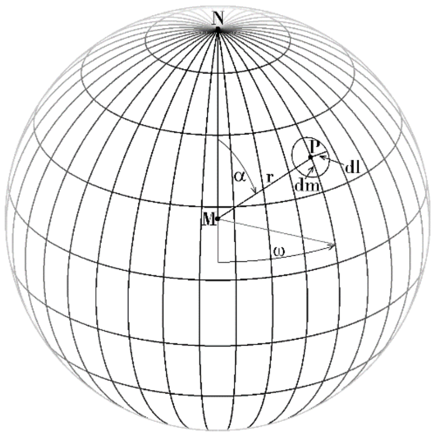

fig. 2

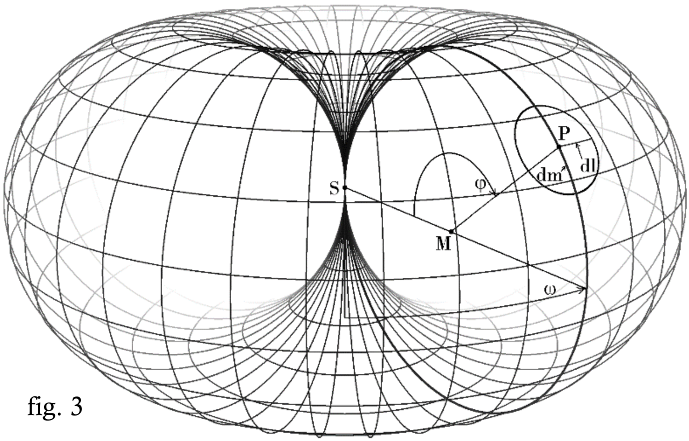

fig. 3

We consider the small circle around any point P and state the condition that the radii dm (in direction of meridians) and dl (parallel to latitudes) have to be equal.

Lengths m of longitudes (meridians) on both figures, sphere and horn torus, are $m = 2\pi \cdot r$ and lengths l of latitudes (computed differently, see supplement for derivation):

$$l = 2\pi \cdot r \cdot \sin\alpha \tag{2.1}$$

on the sphere and, from (5.1)

$$l = 2\pi \cdot r \cdot (1 - \cos\varphi) \tag{2.2}$$

on the horn torus.

The differentials dm - radii of the small circles on the longitude - are

$dm = r \cdot d\alpha$ on the sphere and

$dm = r \cdot d\varphi$ on the horn torus,

differentials dl - radii of the respective small circles on the latitude -

$dl = d\omega \cdot r \cdot \sin\alpha$ on the sphere and

$dl = d\omega \cdot r \cdot (1 - \cos\varphi)$ on the horn torus.

After equalling dm and dl in both figures separately and cancelling r we have for the sphere $d\alpha = d\omega \cdot \sin\alpha$

and for the horn torus $d\varphi = d\omega \cdot (1 - \cos\varphi)$

By solving both equations to $d\omega$ and equalling we get the differential equation

$$d\alpha/\sin\alpha = d\varphi/(1 - \cos\varphi) \tag{2.3}$$

Finally we obtain the condition for conformal mapping by integration:

$$\int(1/\sin\alpha)d\alpha \;=\; \int(1/(1-\cos\varphi))d\varphi \tag{2.4}$$

$$\ln(|\tan(\alpha/2)|) \;=\; -\cot(\varphi/2) + C \tag{2.5}$$

$$\varphi \;=\; 2\cot^{-1}(-\ln(|\tan(\alpha/2)|) - C) \tag{2.6}$$

$$\alpha \;=\; 2\tan^{-1}(e^{-\cot(\varphi/2)\,+\,C}) \tag{2.7}$$

with $0 < \alpha < \pi$, $0 < \varphi < 2\pi$, C any real number.

C is a kind of 'zoom/diminishing factor' for the mapped figures and shifts them: case $\alpha \rightarrow \varphi$: φ moves towards 2π with increasing $C > 0$, towards 0 with $C < 0$, case $\varphi \rightarrow \alpha$: α moves towards π with increasing $C > 0$, towards 0 with $C < 0$, conformality is given for $C \neq 0$ as well, but the mappings are not bijective, when the constant $C \neq 0$ shall be the same in the inverse mapping.

3 Generalised Riemannian Conformal Mapping

The Riemannian stereographic projection (plane $\leftrightarrow$ sphere) likewise is a special case amongst others, the generalised formulas for conformal mapping, replacing (1.1) and (1.2), but then with loss of bijectivity when $C \neq 1$, are

$$\alpha = 2\tan^{-1}(C\cdot|z|) \tag{3.1}$$

$$|z| = \tan(\alpha/2)\;/\;C \tag{3.2}$$

Real number $C > 0$ again is a kind of 'zoom/diminishing factor' for the mapped figures.

4 Mapping from Plane to Horn Torus and vice versa

When we combine (1.2) with (2.6) resp. (2.7) we get the formulas for conformal mapping direct from plane to horn torus and vice versa:

$$\varphi = 2\cot^{-1}(-\ln(|z|) - C) \tag{4.1}$$

$$|z| = e^{-\cot(\varphi/2)\,+\,C} \tag{4.2}$$

5 Supplement: Length of Horn Torus Latitude

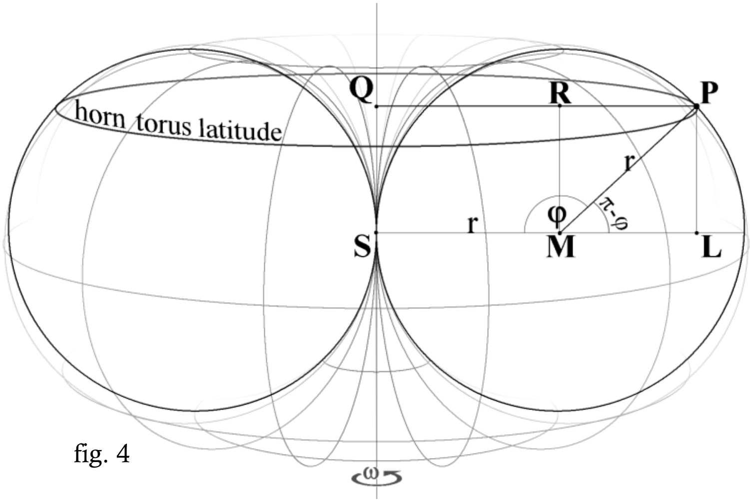

fig. 4

The sketch shows details of a horn torus cross section, embedded in a slightly tilted perspective view, point P is positioned on longitude ω (rotation angle) 90° and latitude φ (torus bulge revolution angle) 135°, S is centre of horn torus, M is centre of circle (half of cross section) with radius r, Q is centre of selected latitude through P and lies on main symmetry axis, auxiliary line $\overline{MR}$ is perpendicular to $\overline{QP}$, auxiliary line $\overline{PL}$ perpendicular to diameter of horn torus cross section circle through S and M. With these points and parameters, we easily can calculate length l of the latitude with radius $\overline{QP}$:

$$\begin{aligned}\overline{QP} &= \overline{QR} + \overline{RP}\\ &= \overline{SM} + \overline{ML}\\ &= r + r\cdot\cos(\pi - \varphi)\\ &= r - r\cdot\cos\varphi\\ &= r\cdot(1 - \cos\varphi)\end{aligned}$$

$$l = 2\pi\cdot\overline{QP}$$

$$l = 2\pi\cdot r\cdot(1 - \cos\varphi) \tag{5.1}$$

6 Addendum 1: Properties of the Horn Torus

For centuries science ignored the existence of the geometric figure horn torus completely or neglected its relevance unduly. Nearly nobody realized, described and applied the exceptional topology, maximum symmetry, high complexity and creative capabilities of this unique object. Now, in recent years, it increasingly appears in publications, mostly in context of particle and quantum physics and in connection to cosmological questions. The subject is developing and promises to stay exciting for a good while.

Horn torus properties then get most thrilling when we add dynamic to the figure, when we let it turn, around the main symmetry axis and around the torus bulge, when we let it change its size during the turns, when we combine all these motions and when we finally interlace two, more or even infinitely many such dynamic horn tori with one another.

As static figure the horn torus is a rather simple object with only little properties, e.g. one can establish the parametric form in three dimensions

$$
\begin{aligned}
x &= r\cdot(1 - \cos\varphi)\cdot\sin\omega \\
y &= r\cdot(1 - \cos\varphi)\cdot\cos\omega \\
z &= r\cdot\sin\varphi
\end{aligned}
$$

or similar, depending on definition of axes and angles (compare fig. 3+4), and the surface area and the volume can be computed as

$$
\begin{aligned}
A &= 4\pi^2 r^2 \\
V &= 2\pi^2 r^3
\end{aligned}
$$

As the horn torus is a special case of ring torus with radii R and r, all findings from there can be taken for the horn torus by equalling the radii, but all that is not really notably.

The topology however differs eminently, compared with the ring torus, and it should be reviewed thoroughly. In respect of the mapping from the Riemann sphere one realizes that on the horn torus the points for zero and for infinity merge to one single point, what induces that strange topology. This property is used by Professor S. Saitoh in connection with his results in the division by zero issue, e.g. in our common recent paper researchgate.net/publication/335463208 and in other publications.

7 Addendum 2: Relevance of Horn Tori

To anticipate it first: The horn torus helps to understand our 'real world'. But note: horn tori are not real, they don't occur as figures in reality, regardless how you define reality. If at all, I myself only talk about very restricted epistemological meanings of the term, but here I do not intend to add any more useless contributions to the general philosophical debate concerning reality, only the remark that all of our many models of nature are crutches to hobble around the never reachable and imaginable truth, solely to enjoy the short, rare moments of supposed close encounter of it.

No actual model of nature comprises all aspects of experienceable reality. We indeed possess lots of mathematical descriptions for all single realized physical processes and apply them very successfully, but no one describes all in one big idea, and furthermore there is a lack of appropriate images that illustrate objects and processes in a way that they match the human thought structure and our mental capability.

Here starts the relevance of horn tori. The associated model, described on the web link below, tries to fill this lack of imagery, what most of us need for a good understanding. It's not a theory but an allegoric representation. The consistent theory, based on this horn torus model, is a current project.

The dynamised horn torus illustrates in an easily intelligible pictorial way the big mysteries in our comprehension of reality:

> intrinsicality of time, space and physical objects,
> continuum vs. discrete nature of space, metrisation of space,
> origin and cause for quantization, minimum values,
> constancy of lightspeed, maximum value,
> non-locality of quantum processes, entanglement,
> arrow of time, determinism, causality, …

The author of this present paper provides a lot of thoughts, texts, images and animations concerning the horn torus model as a kind of intellectual game, first to exercise imaginative power, to improve abstract thinking and to generate lots of aha moments, then as proposal for a different approach to physical questions, and finally as part of an interdisciplinary art project on his private website:

https://www.horntorus.com

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